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VARIATSION HISOBNING ASOSIY MASALASINING LEJANDR HAMDA YAKOBI SHARTI

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A R T I C L E I N F O.

Tayanch iboralar: ikkinchi variatsiya, Lejandr sharti, qo'shib olingan variatsion masala, Yakobi sharti, kuchaytirilgan Yakobi va Lejandr shartlari, Veyershtrass funksiyasi, Veyershtrass sharti, kuchli ekstremumning yetarli sharti, Veyershtrass-Erdman shartlari, kvadratik funksionalning ekstremumi.

Annotatsiya

Ushbu maqolada variatsion hisob asosiy masalasi uchun Lejandr va Yakobi shartlari haqida fikr yuritib, ushbu masalaning nazariy ahamiyatini yoritish.

Ushbu maqolada Variatsion hisob ekstremumining ikkinchi tartibli zaruriy shartlari va yetarli shartlari qaraladi.

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx \qquad (1)$$

funksionalning

$$V = \{y = y(x) \in C^{(1)}[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\} \qquad (2)$$

to'plamdagi ekstremumini topish masalasi, ya'ni variatsion hisob asosiy masalasi, berilgan bo'lsin. Bu yerda $F(x, y, y')$ funksiyani R^3 ning biror ochiq Q to'plamida aniqlangan, $P_0(x_0,y_0)$ va $P_1(x_1,y_1)$ nuqtalarni esa, $S = \{(x, y) : (x, y, z) \in Q\}$ to'plamga tegishli, deb hisoblaymiz.

Lejandr sharti. $y^0=y^0(x)$ joyiz funksiya bo'lsin $(y^0 \in V)$. Shu nuqtada (1) funksionalning ikkinchi variatsiyasini hisoblaymiz. Ta'rifga ko'ra, bu variatsiya,

$$\delta^2 J[y^2, h] = \frac{d^2 J[y^0 + \alpha h]}{d\alpha^2} \Big/_{\alpha=0}$$

formula bo'yicha hisoblanadi, bu yerda

$$h = h(x) \in C^{(1)}[x_0, x_1], h(x_0) = h(x_1) = 0 .$$

Agar $F(x, y, y') \in C^{(2)}(Q)$ deb faraz qilsak, $\varphi(\alpha) = J[y^0 + \alpha h]$ funksiya $\alpha=0$ nuqta atrofida uzluksiz ikkinchi tartibli hosilaga ega. Demak,

$$\delta^2 J[y^0,h] = \frac{d^2}{d\alpha^2}\int_{x_0}^{x_1} F(x, y^0(x)+\alpha h(x), y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0} =$$

$$= \int_{x_0}^{x_1} \frac{\partial^2}{\partial\alpha^2} F(x, y^0(x)+\alpha h(x), y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0} =$$

$$= \int_{x_0}^{x_1} [F_{yy}(x,y^0(x),y^{0'}(x))h^2(x) + 2F_{yy'}(x,y^0(x),y^{0'}(x))h(x)h'(x) + F_{y'y'}(x,y^0(x),y^{0'}(x))h'^2(x)]dx,$$

$$h = h(x) \in C^{(1)}[x_0,x_1],\ h(x_0) = h(x_1) = 0$$

1-teorema (Lejandr). $F(x,y,y') \in C^{(2)}(Q)$ bo'lsin. Agar $y^0(x) \in C^{(1)}[x_0,x_1] - (1)$ funksionalning (2) to'plamdagi kuchsiz minimali (maksimali) bo'lsa,

$$F_{y'y'}(x,y^0(x),y^{0'}(x)) \geq 0 \quad (\leq 0), \quad \forall x \in [x_0,x_1] \tag{3}$$

tengsizlik bajariladi. (3) munosabatga Lejandr sharti deyiladi.

I s b o t i. Faraz qilaylik, (3) bajarilmasin, ya'ni biror $\xi\in[x_0,x_1]$ uchun,

$$F_{y'y'}(x,y^0(x),y^{0'}(x)) < 0 \tag{4}$$

bo'lsin. $F_{y'y'}(x, y^0(x), y^{0'}(x))$ Funksiyaning uzluksizligiga asosan, $\xi\in(x_0,x_1)$ deb olish mumkin. Bundan tashqari, bu funksiyaning uzluksizligidan va (4) tengsizlikdan, shunday $\varepsilon>0$ sonning mavjudligi kelib chiqadiki,

$$\sup F_{y'y'}(x,y^0(x),y^{0'}(x)) = b < 0,\ x \in (x-\varepsilon, x+\varepsilon) \tag{5}$$

bo'ladi. Quyidagi,

$$h_\varepsilon(x) = \begin{cases} \sin^2\dfrac{\pi(x-\xi+\varepsilon)}{2\varepsilon}, & x \in (\xi-\varepsilon, \xi+\varepsilon) \\ 0, & x \notin (\xi-\varepsilon, \xi+\varepsilon) \end{cases}$$

funksiyani qaraymiz, bu funksiya $C^{(1)}[x_0,x_1]$ ga tegishli va

$$h_\varepsilon'(x) = \begin{cases} \dfrac{\pi}{2\varepsilon}\sin\dfrac{\pi(x-\xi+\varepsilon)}{\varepsilon}, & x \in (\xi-\varepsilon, \xi+\varepsilon) \\ 0, & x \notin (\xi-\varepsilon, \xi+\varepsilon) \end{cases}$$

Endi (3) formulada $h = h_\varepsilon(x), h' = h_\varepsilon'(x)$ deb olib, quyidagiga ega bo'lamiz:

$$\delta^2 J[y^0,h_\varepsilon]=\int_{\xi-\varepsilon}^{\xi+\varepsilon}[F_{yy}(x,y^0(x),y^{0'}(x))\sin^4\frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}+$$

$$+F_{yy'}(x,y^0(x),y^{0'}(x))\frac{\pi}{\varepsilon}\sin^2\frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}\sin\frac{(x-\xi+\varepsilon)}{\varepsilon}+$$

$$+F_{y'y'}(x,y^0(x),y^{0'}(x))\frac{\pi^2}{4\varepsilon^2}\sin^2\frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}]dx\le$$

$$\le\frac{1}{\varepsilon^2}\int_{\xi-\varepsilon}^{\xi+\varepsilon}\left[\beta\frac{\pi^2}{4}\sin^2\frac{\pi(x-\xi+\varepsilon)}{\varepsilon}+\varepsilon\pi F_{yy'}(x,y^0(x),y^{0'}(x))\sin^2\frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}\sin\frac{\pi(x-\xi+\varepsilon)}{\varepsilon}+\right]$$

$$+\varepsilon^2F_{yy}(x,y^0(x),y^{0'}(x))\sin^4\frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}]dx$$

Bu yerdan (5) ni e'tiborga olib, yetarli kichik ε>0 uchun $\delta^2 J[y^0,h_\varepsilon]$<0 tengsizlikni olamiz. Ammo, lokal minimumning zaruriy shartiga ko'ra, $\delta^2 J[y^0,h_\varepsilon]$≥0 munosabat bajarilishi kerak. Bu qarama-qarshilik, teoremani minimum uchun isbotlaydi. U maksimum uchun ham, shunga o'xshash isbotlanadi.

Yakobi sharti. Yuqorida isbotlangan Lejandr sharti, lokal minimum (maksimum)ning, funksional ikkinchi variatsiyasi yordamida ifodalanadigan,i $\delta^2 J[y^0,h]\ge 0\ \ (\le 0)$ shartidan foydalanib keltirib chiqariladi. Funksional ikkinchi variatsiyasining ekstremum nuqtasida ishorasini saqlashini ifodalovchi bu shartdan yana bitta ikkinchi tartibli zaruriy shart-Yakobi shartini keltirib chiqarish mumkin.

$F(x,y,y')\in C^{(2)}(Q)$ deb hisoblab, $y^0(x)$ joyiz funksiya uchun,

$$w(x,h,h')=F_{y'y'}(x,y^0(x),y^{0'}(x))h'^2+2F_{yy'}(x,y^0(x),y^{0'}(x))hh'+F_{yy}(x,y^0(x),y^{0'}(x))h^2$$ funksiyani qaraymiz. U vaqtda, (3) formulaga ko'ra,

$$\delta^2 J[y^0,h]=\int_{x_2}^{x_1}\omega(x,h,h')dx \qquad (6)$$

bo'ladi. Agar $y^0(x)$ kuchsiz minimal (maksimal) bo'lsa, $\delta^2 J[y^0,h]\ge 0\ \ (\le 0)$ shart barcha $h(x)\in C^{(1)}[x_0,x_1]$, $h(x_0)=h(x_1)=0$ funksiyalar uchun bajariladi. $h^0(x)=0$ uchun esa $\delta^2 J[y^0,h^0]=0$ bo'lishi ravshan. Demak, qaralayotgan variatsion hisob masalasiga, *qo'shib olingan ekstremal masala* deb ataluvchi,

$$\left.\begin{aligned}&\delta^2 J[y^0,h]=\int_{x_0}^{x_1}\omega(x,h,h')dx\to\min(\max)\\&\mathrm{h(x_0)=h(x_1)=0,\ \ h(x)\in C^{(1)}[x_0,x_1]}\end{aligned}\right\} \qquad (7)$$

Masala, $h^0(x)=0$ yechimga ega.

Faraz qilaylik, $F(x,y,y')\in C^{(3)}(Q), y^0(x)\in C^{(2)}[x_0x_1]$ - joyiz stasionar funksiya $F_{y'y'}(x,y^0(x),y^{0'}(x))\ne 0\ \forall x\in[x_0x_1]$ bo'lsin. U vaqtda, (7) masala uchun tuzilgan,

$$\omega_h(x,h,h')-\frac{d}{dx}\omega_{h'}(x,h,h')=0$$

Eyler tenglamasiga, variason hisob asosiy masalasi uchun *Yakobi tenglamasi* deyiladi. $\omega(x,h,h')$ funksiyaning ko'rinishini hisobga olib, Yakobi tenglamasini

$$A(x)h'' + B(x)h' + C(x)h = 0 \qquad (8)$$

ikkinchi tartibli bir jinsli chiziqli differensial tenglama ko'rinishida yozish mumkin, bu yerda

$$A(x) = F_{y'y'}(x, y^0(x), y^{0'}(x)),$$

$$B(x) = \frac{d}{dx}F_{y'y'}(x, y^0(x), y^{0'}(x)), C(x) = \frac{d}{dx}F_{yy'}(x, y^0(x), y^{0'}(x)) - F_{yy}(x, y^0(x), y^{0'}(x))$$

Differensial tenglamalar kursidan ma'lumki, (8) tenglama $h(x_0) = 0,\ h'(x_0) = 1$ chegaraviy shartlarni qanoatlantiruvchi (aynan noldan farqli) yagona yechimga ega. Shu yechimning x_0 dan farqli nollariga, *x_0 nuqtaga qo'shma nuqta* deyiladi. Qo'shma nuqtaga yana quyidagi ekvivalent ta'rifni ham berish mumkin.

T a' r i f. Agar (8) Yakobi tenglamasi $h(x_0) = 0,\ h(x^*) = 0 \quad x^* \neq x_0$ shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan $h(x),\ x \in [x_0, x_1]$ yechimga ega bo'lsa, x^* nuqtaga $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga qo'shma nuqta deyiladi.

<u>2-t e o r e m a (Yakobi).</u> Faraz qilaylik:

a) $F(x, y, y') \in C^{(3)}(Q)$, б) $y^0(x) \in C^{(2)}[x_0 x_1]$ - kuchsiz minimal (maksimal)

$F_{y'y'}(x, y^0(x), y^{0'}(x)) > 0 \quad (< 0) \quad \forall x \in [x_0 x_1]$ bo'lsin. U holda, $y^0(x)$ funksiya Yakobi shartini qanoatlantiradi: (x_0,x_1) intervalda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma bo'lgan nuqta mavjud emas.

I s b o t i. Teskarisini faraz qilamiz. $y^0(x),\ x \in [x_0,x_1]$ joyiz chiziq bo'ylab x_0 ga qo'shma bo'lgan $x^* \to x \in [x_0,x_1]$ nuqta mavjud bo'lsin. $h^*(x) \neq 0,\ x \in [x_0,x_1]$ esa, Yakobi tenglamasining unga mos yechimi bo'lsin, ya'ni

$$\omega_h(x, h^*(x), h^{*\prime}(x)) - \frac{d}{dx}\omega_{h'}(x, h^*(x), h^{*\prime}(x)) \equiv 0,\ \forall x \in [x_0, x_1],\ h^*(x_0) = h^*(x^*) = 0 \quad (9)$$

shartlar bajarilsin. Qo'shma nuqta ta'rifiga ko'ra, $h^{*'}(x^* - 0) \neq 0$ shart bajariladi.Quyidagi

$$h(x) = \begin{cases} h^*(x), x \in [x_0, x^*], \\ 0, x \in (x^*, x_1]. \end{cases} \qquad (10)$$

funksiyani tuzamiz.

Endi $2\omega(x,h,h') = h\omega_h + h'\omega_{h'}$ formulani, hamda (10), (11) larni hisobga olib, (8) dan quyidagini olamiz:

$$d^2J[y^0, h] = \int_{x_0}^{x_1} \omega(x,h,h')dx = \int_{x_0}^{x^*} \omega(x,h,h')dx = \frac{1}{2}\int_{x_0}^{x^*} (h^*\omega_h + h^{*\prime}\omega_{h'})dx =$$

$$\frac{1}{2}\int_{x_0}^{x^*} (h^*\frac{d}{dx}\omega_h + h^{*\prime}\omega_{h'})dx = \frac{1}{2}\int_{x_0}^{x^*} \frac{d}{dx}(h^*\omega_{h'})dx = \frac{1}{2}h^*(x)\omega_h(x, h^*(x), h^{*\prime}(x))\Big|_{x_0}^{x^*} = 0.$$

Shunday qilib, (10) funksiya - (7) masalaning yechimidir. teng munosabatlar ko'rsatadiki, h(x) funksiya uchun x^* - bukilish nuqtasidir. Natijida, $x=x^*$ nuqtada

$$\omega_h(x,h,h')|_{x=x^*-0} = \omega_{h'}(x,h,h')|_{x=x^*+0} \tag{11}$$

Veyershtrass - Erdman sharti bajariladi. $\omega(x,h,h')$ funksiyaning ko'rinishini hisobga olib, (12) tenglikni quyidagicha yozamiz:

$$\begin{aligned}&[h(x)F_{yy}(x,y^0(x),{y^0}'(x))+h'(x)F_{y'y'}(x,y^0(x),{y^0}'(x))]|_{x=x^*-0}=\\&=[h(x)F_{yy}(x,y^0(x),{y^0}'(x))+h'(x)F_{y'y'}(x,y^0(x),{y^0}'(x))]|_{x=x^*+0}\end{aligned} \tag{12}$$

$h(x^*-0)=h(x^*+0)=0$ va $F_{y'y'}(x,y^0(x),{y^0}'(x))$ uzluksiz bo'lgani uchun, (12) tenglik

$$[h'(x^*-0)-h'(x^*+0)]F_{y'y'}(x^*,y^0(x^*),{y^0}'(x^*))=0 \tag{13}$$

ko'rinishga keladi. Teoremaning shartiga ko'ra $F_{y'y'}(x^*,y^0(x^*),{y^0}'(x^*))\neq 0$. U vaqtda (13) dan $h'(x^*-0)=h'(x^*+0)=0$ bo'lishi kelib chiqadi. Bu esa x^* ning bukilish nuqtasi ekanligiga ziddir. Olingan qarama-qarshilik teoremani isbotlaydi.

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