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A MIXED PROBLEM FOR A FOURTH ORDER TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATION WITH TWO LINES OF DEGENERACY

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Abstract

This article examines an initial-boundary value problem for a fourth-order partial differential equation with a boundary condition violation. By using the method of separation of variables, the problem is reduced to a spectral problem for an ordinary differential equation dependent on the spatial variable. Then, this spectral problem is expressed in an equivalent form as an integral equation with a symmetric kernel. Utilizing the theory of integral equations with symmetric kernels, the eigenvalues, eigenfunctions, and the convergence of the corresponding series are proven. The solution to the problem is constructed as a series in terms of the eigenfunctions of the spectral problem. Necessary conditions for the uniqueness of the solution are determined.

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1. INTRODUCTION

The study of differential equations has a deep and extensive history. Until the late 20th century, most research concentrated on equations involving derivatives of integer order. However, with the development of fractional calculus which generalizes differentiation and integration interest began to grow in differential equations with fractional derivatives. Currently, a significant portion of research is dedicated to solving initial, boundary, and spectral problems for both ordinary and partial differential equations that involve different types of fractional derivatives (see, for example, [1]-[4] and [5], as well as the references therein). Additionally, the books [6] and [7] have played a crucial role in advancing this area of study.

We provide a brief summary of research related to the topic of this paper, which focuses on a fourth-order partial differential equation featuring a fractional time derivative of an unknown function.

Several studies, including [8]-[10], examine initial-boundary value problems for fourth-order equations in one and two spatial dimensions, using the Caputo fractional time derivative. Notably, [10] also explores the associated inverse problem. In [11], [12], and [13], the authors study initial-boundary value problems for fourth-order equations involving different types of fractional derivatives, such as the Hilfer, Dzhribashyan-Nersesyan, and Riemann-Liouville derivatives. Additionally, direct and inverse problems for a mixed-type fourth-order equation involving the Hilfer derivative are investigated in [14] and [15], respectively. Especially noteworthy are studies [17] and [18], which address inverse problems aimed at determining the order of the fractional derivative in the Riemann-Liouville and Caputo

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frameworks.

The studies mentioned above primarily focus on non-degenerate equations. In contrast, both local and nonlocal boundary value problems for degenerate partial differential equations involving fractional derivatives of an unknown function have not been thoroughly explored. Initial-boundary value problems for degenerate fourth-order equations with time-fractional derivatives, as well as first- and second-order time derivatives, have been addressed in earlier works such as [18]-[20].

In the present paper, we study a mixed problem for a fourth-order partial differential equation with two lines of degeneracy.

2. FORMULATION OF PROBLEM

In a rectangular domain $\Omega=\{(x,t):0<x<1,0<t<T\}$, we consider the following equation

$$ {}_{C}D_{0t}^{\alpha}u(x,t)+\left[x^{\beta}(1-x)^{\gamma}u_{xx}(x,t)\right]_{xx}=f(x,t), \tag{1}$$

where ${}_{C}D_{0t}^{\alpha}$ is Caputo fractional differential operator α order [1]

$$ {}_{C}D_{0t}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{g^{(n)}(z)dz}{(t-z)^{\alpha-n+1}},\qquad (n=\mathrm{Re}[\alpha]+1,t>0)$$

α, β,γ, T are given real numbers, such that $1<\alpha<2$, $0<\beta,\gamma<1$, $T>0$; $f(x,t)$ is a given function on Ω, $\Gamma(\cdot)$ is Euler's gamma function.

Definition 1. A function $u(x,t)$ satisfying equation (1) in domain Ω, and the following conditions $u(x,t)\in C(\bar{\Omega})$, ${}_{C}D_{0t}^{\alpha}u(x,t)$, $\left(x^{\beta}(1-x)^{\gamma}u_{xx}\right)_{xx}\in C(\Omega)$ is called regular in the domain Ω solution of the equation (1).

Problem 1. Find a regular in the domain Ω solution of the equation (1) satisfying the following initial

$$ u(x,0)=\varphi(x),\ u_{t}(x,0)=\psi(x),\ 0\le x\le 1 \tag{2}$$

and the boundary conditions:

$$ u(1,t)=0,\ u_{x}(1,t)=0,\ \lim_{x\to 0}x^{\beta}(1-x)^{\gamma}u_{xx}(x,t)=0$$

$$ \lim_{x\to 0}\left[x^{\beta}(1-x)^{\gamma}u_{xx}(x,t)\right]_{x}=0,\ 0<t\le T, \tag{3}$$

where $\varphi(x)$, $\psi(x)$ are sufficiently smooth given functions, such that $\varphi(1)=0,\varphi'(1)=0$, $\lim_{x\to 0}x^{\beta}(1-x)^{\gamma}\varphi''(x)=0$, $\lim_{x\to 0}\left[x^{\beta}(1-x)^{\gamma}\varphi''(x)\right]'=0$.

First, we consider homogeneous equation corresponding to the equation (1), and seek a solution of this homogeneous equation satisfying the condition (3) in the form $u(x,t)=v(x)T(t)$. As a result, with respect to $v(x)$, we have the following spectral problem:

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(4) $$\left[x^{\alpha}(1-x)^{\beta}v''(x)\right]'' = \lambda v(x), \qquad x \in (0,1),$$

(5) $$v(1)=0,\ v'(1)=0,\ \lim_{x\to 0} x^{\beta}(1-x)^{\gamma}v''(x)=0,\ \lim_{x\to 0}\left[x^{\beta}(1-x)^{\gamma}v''(x)\right]'=0,$$

i.e. a problem finding those values of the parameter λ for which there exist nontrivial solution of equation (4) satisfying conditions (5).

Let's investigate the problem {(4),(5)}. Assume that there exist eigenvalues of the spectral problem {(4),(5)}. Considering this assumption, first, we will define the sign of the λ. To do this, multiplying the both sides of (4) to the function $v(x)$ and integrating over $[0,1]$, and using the rule of integration by parts two times, considering the condition (5), we have

$$\lambda\int_0^1 v^2(x)\,dx = \int_0^1 x^{\beta}(1-x)^{\gamma}\left[v''(x)\right]^2 dx,$$

from which it follows that $\lambda \geq 0$.

Let $\lambda = 0$. Then, form (1), we have $\left[x^{\beta}(1-x)^{\gamma}v''(x)\right]'' = 0$. It is easy to see that the general solution of this equation has the following form

$$v(x) = C_1\int_0^x (x-z)z^{1-\beta}(1-z)^{-\gamma}\,dz + C_2\int_0^x (x-z)z^{-\beta}(1-z)^{-\gamma}\,dz + C_3x + C_4,$$

where C_j, $j=\overline{1,4}$ are arbitrary constants. Satisfying this function to the conditions (5), it is easy to verify that $v(x)\equiv 0$. Thus, we conclude that the problem {(4),(5)} can have nontrivial solutions only for $\lambda > 0$.

Now, we will show the existence of the eigenvalues and eigenfunctions of the problem {(4),(5)}. In order to this, we construct Green's function of the problem {(4),(5)}. It is unique and represents as follows:

$$G(x,s)=\begin{cases}\displaystyle\int_0^s (s-z)z^{1-\beta}(1-z)^{-\gamma}\,dz - x\int_0^s (s-z)z^{-\beta}(1-z)^{-\gamma}\,dz + \\ \displaystyle +s\int_0^1 (x-z)z^{-\beta}(1-z)^{-\gamma}\,dz - \int_0^1 (x-z)z^{1-\beta}(1-z)^{-\gamma}\,dz, & x<s,\\ \displaystyle\int_0^x (x-z)z^{1-\beta}(1-z)^{-\gamma}\,dz - s\int_0^x (x-z)z^{-\beta}(1-z)^{-\gamma}\,dz + \\ \displaystyle +x\int_0^1 (s-z)z^{-\beta}(1-z)^{-\gamma}\,dz - \int_0^1 (s-z)z^{1-\beta}(1-z)^{-\gamma}\,dz, & x>s.\end{cases}$$

Then the problem {(4),(5)} will be equivalent to the integral equation [1]

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(6) $$v(x)=\lambda\int_0^1 G(x,s)v(s)ds.$$

Since $G(x,s)$ is symmetric, continuous, and positive ($\lambda>0$), then from the theory of integral equations with symmetric kernels [21] it follows that equation (6) [hence problem {(4),(5)}] has a countable set of eigenvalues $0<\lambda_1<\lambda_2<\lambda_3<...$ and corresponding orthonormal eigenfunctions $v_1(x),v_2(x),v_3(x),...$ and, also any function $g(x)\in L_2[0,1]$ expands into a series in these eigenfunctions, which converges on average.

3. CONVERGENCE OF BASIC BILINEAR SERIES

Let $\{v_k(x)\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(4),(5)}, respectively. We proved the following lemmas in our previous work [23].

Lemma 1. On the interval $[0,1]$ the following series converge uniformly:

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[x^{\beta}(1-x)^{\gamma}v_k''(x)\right]^2}{\lambda_k^2},\ \sum_{k=1}^{+\infty}\frac{\left\{\left[x^{\beta}(1-x)^{\gamma}v_k''(x)\right]'\right\}^2}{\lambda_k^2}.$$

Lemma 2. Let the following conditions be fulfilled*:*

$$f(x),\ f'(x)\in C[0,1],\ x^{\beta/2}(1-x)^{\gamma/2}f''(x)\in L_2[0,1],\ f(1)=0,\ f'(1)=0.$$

Then the following Bessel type inequality holds:

(7) $$\sum_{k=1}^{\infty}\lambda_k f_k^2\le\int_0^1 x^{\beta}(1-x)^{\gamma}\left[f''(x)\right]^2dx,$$

here in after f_k denotes Fourier coefficient of the function $f(x)$ by the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$.

Lemma 3. Suppose that the following conditions hold:

$$f(x),\ f'(x),\qquad x^{\beta}(1-x)^{\gamma}f''(x),\qquad \left[x^{\beta}(1-x)^{\gamma}f''(x)\right]'\in C[0,1],$$

$$\left[x^{\beta}(1-x)^{\gamma}f''(x)\right]''\in L_2[0,1],$$

$$f(1)=0\ ,\ f'(1)=0,\ \lim_{x\to 0}x^{\beta}(1-x)^{\gamma}f''(x)=0,\ \lim_{x\to 0}\left[x^{\beta}(1-x)^{\gamma}f''(x)\right]'=0.$$

Then the following Bessel type inequality holds:

(8) $$\sum_{k=1}^{\infty}\lambda_k^2 f_k^2\le\int_0^1\left[\left(x^{\beta}(1-x)^{\gamma}f''(x)\right)''\right]^2dx.$$

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4. THE MAIN RESULTS

Theorem 1. Suppose that the following conditions hold:

1. $\varphi(x)$, $\psi(x)$ satisfy conditions of the lemma 2.
2. $f(x,t)$ satisfies conditions of the lemma 2 uniformly with respect to t
3. $f_t'(x,t)$ satisfies conditions of the lemmas 2 and 3 uniformly with respect to variable t.

Then the following

$$u(x,t)=\sum_{k=1}^{+\infty}\Big[\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)+\psi_k t E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)+$$

$$+\int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^{\alpha}\right] f_k(z)\,dz\Big] v_k(x) \tag{9}$$

series will be a regular solution of the problem $\{(1),(2),(3)\}$,where $E_{\alpha,\beta}(z)$ is two-parameter Mittage-Leffler's function [22]:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}.$$

Proof. We seek solution of the considered in the form

$$u(x,t)=\sum_{k=1}^{\infty}T_k(t)v_k(x), \tag{10}$$

where $v_k(x)$, $k\in N$ are functions which are defined by (6), and $T_k(t)$, $k\in N$ are unknown functions.

Expanding functions $f(x,t)$ and $\varphi(x)$ to series by the functions $\{v_k(x)\}_{k=1}^{+\infty}$, we get

$$f(x,t)=\sum_{k=1}^{\infty}f_k(t)v_k(x),\quad \varphi(x)=\sum_{k=1}^{\infty}\varphi_k v_k(x),\ \psi(x)=\sum_{k=1}^{\infty}\psi_k v_k(x), \tag{11}$$

where $f_k(t)=\int_0^1 f(x,t)v_k(x)\,dx$, $\varphi_k=\int_0^1\varphi(x)v_k(x)\,dx$, $\psi_k=\int_0^1\psi(x)v_k(x)\,dx$.

Substituting (10) and (11) into equation (1) and the condition (2), by virtue of (8) and completeness of the system of functions $v_k(x)$, $k\in N$ in $L_2(0,1)$, it follows the following problem with respect to unknown function $T_k(t)$:

$${}_C D_{0t}^{\alpha}T_k(t)+\lambda_k T_k(t)=f_k(t),\ T_k(0)=\varphi_k,\qquad T_k'(0)=\psi_k\ .$$

It is known the solution of this problem represented as follows [22]

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$$T_k(t)=\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)+\psi_k t E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)+\int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^{\alpha}\right] f_k(z)dz.$$

Substituting this expression of $T_k(t)$ into (10), we obtain formula (9).

Now, we show uniformly convergence of the series

$$u(x,t),\ u_x(x,t),\ x^{\beta}(1-x)^{\gamma}u_{xx}(x,t),\ \left[x^{\beta}(1-x)^{\gamma}u_{xx}(x,t)\right]_x,\ {}_C D_{0t}^{\alpha}u(x,t).$$

Using the rule of integrating by parts we rewrite (9) as follows

$$u(x,t)=\sum_{k=1}^{+\infty}\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)v_k(x)+t\sum_{k=1}^{+\infty}\psi_k E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)v_k(x)+$$

$$+t^{\alpha}\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)f_k(0)v_k(x)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz v_k(x). \tag{12}$$

Since Mittag-Leffler function is bounded [22], i.e.

$$\left|E_{\alpha,\beta}(-z)\right|\le M,\ 0<M<+\infty,\ z>0, \tag{13}$$

one can easily verify that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)v_k(x)\right|\le C_1\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$t\sum_{k=1}^{+\infty}\left|\psi_k E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)v_k(x)\right|\le C_2\left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}f_k(0)t^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)v_k(x)\le C_3\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\cdot v_k(x)\le C_4\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

where $C_1=M$, $C_2=MT$, $C_3=MT^{\alpha}$, $C_4=M\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}$.

We show the validity of the first inequality. Using (13) and Cauchy Schwarz inequality, we obtain

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$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k(x)\right| \le M\sum_{k=1}^{+\infty}\left|\varphi_k v_k(x)\right| = C_1\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k}\varphi_k\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \le C_1\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

Similarly, one can show validity of the second and third inequalities.

From these inequalities, considering Lemma 1 and Lemma 2, it follows that absolutely and uniformly convergence of the series of (9) and $u(x,t)\in C\left(\bar{\Omega}\right)$.

Differentiating (9) with respect to x, we obtain

$$u_x(x,t) = \sum_{k=1}^{+\infty}\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)v_k'(x) + t\sum_{k=1}^{+\infty}\psi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)v_k'(x) +$$

$$+t^{\alpha}\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)f_k(0)v_k'(x) +$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\,v_k'(x). \tag{14}$$

Similarly, it can be shown that the following inequalities hold:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k'(x)\right| \le C_1\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$t\sum_{k=1}^{+\infty}\left|E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)_k\psi_k v_k'(x)\right| \le C_2\left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$t^{\alpha}\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)f_k(0)v_k'(x)\right| \le C_3\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^{\alpha}\right]f_k'(z)dz\,v_k'(x) \le$$

$$\le C_4\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\sum_{k=1}^{+\infty}\frac{v_k'^2(x)}{\lambda_k}\right]^{1/2}.$$

From the last inequalities by virtue of the conditions of Theorem 1, it follows that the absolute and uniformly convergence of the series in (14).

Now, differentiating (14) with respect to x and multiplying the both sides of the taken equality to $x^{\beta}(1-x)^{\gamma}$, we obtain

$$x^{\beta}(1-x)^{\gamma}u_{xx}(x,t) = \sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}(1-x)^{\gamma}v_k''(x) +$$

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$$+t\sum_{k=1}^{+\infty}E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right)\psi_k x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)+t^{\alpha}\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right)f_k\left(0\right)x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t\left(t-z\right)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k\left(t-z\right)^{\alpha}\right]f_k'\left(z\right)dzx^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right). \quad (15)$$

We show the convergence of the series in (15). Firstly, we show that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right|\le C_1\left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_k^2\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$t\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\psi_k x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right|\le C_2\left[\sum_{k=1}^{+\infty}\lambda_k^2\psi_k^2\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right]^2}{\lambda_k^2}\right]^{1/2}$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)t^{\alpha}f_k\left(0\right)x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right|\le C_3\left[\sum_{k=1}^{+\infty}\lambda_k^2 f_k^2\left(0\right)\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t\left(t-z\right)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k\left(t-z\right)^{\alpha}\right]f_k'\left(z\right)dzx^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\le$$

$$\le C_4\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2 f_k'^2\left(z\right)dz\sum_{k=1}^{+\infty}\frac{x^{2\beta}\left(1-x\right)^{2\gamma}v_k''^2\left(x\right)}{\lambda_k^2}\right]^{1/2}.$$

Let's prove the first estimate. Taking into account (13), we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right|\le C_1\sum_{k=1}^{+\infty}\left|\varphi_k x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)\right|=C_1\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_k\frac{x^{\beta}\left(1-x\right)^{\gamma}v_k''\left(x\right)}{\lambda_k}\right|.$$

Applying Cauchy-Schwarz inequality from the last, we obtain proof of the first inequality. The second and third inequalities are proved similarly.

Using the same method one can prove absolute and uniformly convergence of the series in $\left[x^{\beta}\left(1-x\right)^{\gamma}u_{xx}\left(x,t\right)\right]_x$.

Now, we investigate ${}_C D_{0t}^{\alpha}u\left(x,t\right)$. Applying differential operator ${}_C D_{0t}^{\alpha}$ to the both sides of (9) and taking into account ${}_C D_{0t}^{\alpha}T_k\left(t\right)=-\lambda_k T_k\left(t\right)$ and also the view of the function $T_k\left(t\right)$, we find

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$$ {}_C D_{0t}^{\alpha} u(x,t) = -\sum_{k=1}^{+\infty} \lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right) \varphi_k v_k(x) - t \sum_{k=1}^{+\infty} \lambda_k E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right) \psi_k x^{\beta} (1-x)^{\gamma} v_k''(x) - $$

$$ -t^{\alpha} \sum_{k=1}^{+\infty} \lambda_k E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) f_k(0) v_k(x) - \sum_{k=1}^{+\infty} \lambda_k \int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z) dz v_k(x). $$

And convergence of the series in ${}_C D_{0t}^{\alpha} u(x,t)$ follows by the following inequalities

$$ \sum_{k=1}^{+\infty} \left| \lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right) \varphi_k v_k(x) \right| \le C_1 \left[\sum_{k=1}^{+\infty} \lambda_k^3 \varphi_k^2 \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2}, $$

$$ t \sum_{k=1}^{+\infty} \left| \lambda_k E_{\alpha,2}\left(-\lambda_k t^{\alpha}\right) \psi_k v_k(x) \right| \le C_2 \left[\sum_{k=1}^{+\infty} \lambda_k^3 \psi_k^2 \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2} $$

$$ \sum_{k=1}^{+\infty} \left| \lambda_k E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) t^{\alpha} f_k(0) v_k(x) \right| \le C_3 \left[\sum_{k=1}^{+\infty} \lambda_k^3 f_k^2(0) \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2}, $$

$$ \sum_{k=1}^{+\infty} \left| \lambda_k \int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z) dz v_k(x) \right| \le $$

$$ \le C_4 \left[\int_0^T \sum_{k=1}^{+\infty} \lambda_k^3 f_k'^2(z) dz \sum_{k=1}^{+\infty} \frac{v^2(x)}{\lambda_k} \right]^{1/2}. $$

These inequalities are proved the same method as above-proved inequalities. From these inequalities, it follows the absolute and uniform convergence of the series in ${}_C D_{0t}^{\alpha} u(x,t)$.

5. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

Now, we prove the uniqueness of the solution of the problem. For this aim, we introduce the following function

$$ T_k(t) = \int_0^1 u(x,t) v_k(x) dx. \tag{16} $$

Based on (16), we consider the following auxiliary function

$$ T_{k,\varepsilon}(t) = \int_{\varepsilon}^{1-\varepsilon} u(x,t) v_k(x) dx, \tag{17} $$

where ε is sufficiently small positive number.

Applying differential operator ${}_C D_{0t}^{\alpha}$ to (17) and using homogeneous equation corresponding equation (1), we get

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$$ {}_C D_{0t}^{\alpha} T_{k,\varepsilon}(t) = -\int_{\varepsilon}^{1-\varepsilon} \left[x^{\beta}(1-x)^{\gamma} u_{xx}(x,t) \right]_{xx} v_k(x)\,dx . $$

Using the rule of integration by parts four times from the last equality, we get

$$ {}_C D_{0t}^{\alpha} T_{k,\varepsilon}(t) = -\Big\{ v_k(x)\left[x^{\beta}(1-x)^{\gamma} u_{xx}(x,t) \right]_x - v_k'(x) x^{\beta}(1-x)^{\gamma} u_{xx}(x,t) + x^{\beta}(1-x)^{\gamma} v_k''(x) u_x(x,t) - $$

$$ -\left[x^{\beta}(1-x)^{\gamma} v_k''(x) \right]' u(x,t) \Big\} \Big|_{x=\varepsilon}^{x=1-\varepsilon} - \int_{\varepsilon}^{1-\varepsilon} \left[x^{\beta}(1-x)^{\gamma} v_k''(x) \right]'' u(x,t)\,dx . \tag{18} $$

Passing to the limit as $\varepsilon \to 0$ and taking (8) and (16) into account, from (18), we derive

$$ {}_C D_{0t}^{\alpha} T_k(t) + \lambda_k T_k(t) = 0 . \tag{19} $$

From (16), we have

$$ T_k(0) = \int_0^1 u(x,0) v_k(x)\,dx = \varphi_k . \tag{20} $$

It is known that the solution of the equation (19) satisfying condition (20) is represented as follows

$$ T_k(t) = \varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right). $$

Let $\varphi(x) \equiv 0$, $x \in [0,1]$. Then from the last equality taking into account (16) for all $t \in [0,\mathrm{T}]$ and $k \in N$ it follows that

$$ \int_0^1 u(x,t) v_k(x)\,dx = 0 . \tag{21} $$

As since problem {(4),(5)} is self-adjoint, its eigenfunctions will be complete system in $L_2[0,1]$. Taking this account from (21), we get $u(x,t) \equiv 0$ almost everywhere on $[0,1]$ for all $t \in [0,T]$. By virtue of $u(x,t) \in C(\overline{\Omega})$, we obtain $u(x,t) \equiv 0$ in $\overline{\Omega}$. Thus, homogeneous problem has only trivial solution and this give us the uniqueness of the solution of the considered problem.

6. CONCLUSION

In the present paper, we consider an initial-boundary value problem for a fourth-order, timefractional, space-degenerate partial differential equation in a rectangular domain. By applying the method of separation of variables, we obtain a spectral problem for an ordinary differential equation. Since this equation has a degenerate coefficient, we cannot find the eigenvalues and eigenfunctions of this spectral problem in explicit form. However, by applying the theory of integral equations with symmetric kernels, we establish the existence and some properties of the eigenvalues and eigenfunctions of this spectral problem. Using these properties, we prove the existence and uniqueness of the main problem.

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