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GOSPODARKA I INNOWACJE

LABORATORIUM WIEDZY Artur Borcuch

Volume: 64 | 2025

Economy and Innovation
ISSN: 2545-0573

For more information contact: editor@gospodarkainnowacje.pl

UNIQUE SOLVABILITY OF A MIXED PROBLEM FOR A TIME-FRACTIONAL SPACE DEGENERATE BEAM EQUATION

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A R T I C L E I N F O.

Keywords: degenerate equation, a mixed problem, spectral method, Green's function, integral equation with symmetric kernel.

Abstract

In this work, in a rectangular domain, we study a mixed problem for a fourth order differential equation degenerating on the bound of the domain. By applying the method of separation of variables to the considered problem, a spectral problem for an ordinary differential equation has been obtained. Then, the Green's function of the spectral problem has been constructed, with the help of which it is equivalently reduced to the second kind Fredholm integral equation with a symmetric kernel. Using the theory of integral equations with symmetric kernels the existence and some properties of the eigenfunctions and eigenvalues of this spectral problem has been studied. The solution of the original problem has been written as the sum of a Fourier series with respect to the system of eigenfunctions of the spectral problem. Uniformly convergence of this series has been proved.

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I. INTRODUCTION

Higher-order differential equations are essential tools for modeling and analyzing a broad spectrum of physical phenomena, including engineering structures, biological systems, and fluid flows. These equations describe how systems evolve under the influence of forces, energy, and dynamic interactions, and they are especially valuable for capturing complex or nonlinear behaviors that simpler models fail to address. In this setting, the investigation of boundary value problems (BVPs) for higher-order differential equations is crucial for establishing the existence of solutions and interpreting their physical meaning. For further details, see [1], [2], [3], [4], [5], and the references cited therein.

A wide range of problems concerning the vibrations of beams, rods, and plates are central to the study of stability and structural mechanics. Modeling these vibrational processes frequently leads to fourth-order differential equations (see, for example, [6], [3], and the references therein).

Among the different classes of higher-order differential equations, beam equations have attracted considerable attention due to their importance in structural engineering and materials science. It is well known that the vibrations of a dynamic beam are described by the following equation:

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(1) $\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)=-\mu\frac{\partial^2 w}{\partial t^2}+q(x,t)$,

where E is the elastic modulus, I is the second moment of area of the beam's cross section, μ is the linear mass density of the beam, $q(x,t)$ and $w(x,t)$ describe the external load and the deflection of the beam in the z direction at position x and time t, respectively. If the beam is homogeneous then *EI* is not depend on x and the equation (1) takes the following simpler form

(2) $EI\frac{\partial^4 w}{\partial x^4}=-\mu\frac{\partial^2 w}{\partial t^2}+q(x,t)$.

Introducing the notations $\alpha^2=EI/\mu, F(x,t)=q(x,t)/\mu$, from (2) we have

(3) $\frac{\partial^2 w}{\partial t^2}+\alpha^2\frac{\partial^4 w}{\partial x^4}=F(x,t)$.

In recent years, much attention has been given to the study of problems involving equation (3). For example, in [7], A.K. Urinov and M.S. Azizov address the inverse problem of identifying the spacedependent source coefficient $f(x)$ in the following beam equation

$$u_{tt}+u_{xxxx}=Q(t)f(x)$$

in a rectangular domain using Fourier's method, where $Q(t)$ is a given function.

Additionally, in [8], K. Sabitov constructs a solution to the Cauchy problem for equation (3) when $F(x,t)\equiv 0$ in the half-plane. This result was later generalized by S. Sitnik, S. Karimov, and Y. Tulasheva in [9] to a degenerate beam equation of the form

$$u_{tt}+t^p u_{xxxx}+\lambda^2 t^p u=0$$

where $\lambda, p\in$ □ such that $p\geq 0$.

In recent years, there has been increasing interest in exploring various problems associated with the beam equation and its extensions. Significant contributions include the analysis of the unique solvability of initial-boundary value problems in [10], [11], investigations of direct and inverse problems in [12], [13], [14], and studies on blow-up solutions in [15], [16].

Nevertheless, although considerable research has been conducted on beam equations of the form (3) and their time-degenerate variants, boundary value problems for the beam equation in the form of equation (1) have received comparatively little attention.

Additionally, recent research has also turned toward beam equations involving fractional-order operators. For instance, the studies in [17], [18], and related works have made notable contributions to this emerging field.

Based on above mentioned facts, in this paper we aimed to study unique solvability of a mixed problem for the time-fractional analogue of the beam equation with $EI=x^\beta$. More precisely, in a rectangular domain Ω such that $\Omega=\{(x,t): 0<x<1,\quad 0<t\leq T<\infty\}$, we consider the following equation

(4) ${}_C D_{0t}^{\alpha}u(x,t)+\left[x^\beta u_{xx}(x,t)\right]_{xx}=f(x,t)$,

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where ${}_C D_{0t}^{\alpha}$ is the Caputo differential operator of α fractional order, $\alpha, \beta \in$ such that $\alpha \in (1,2)$, $0 < \beta < 1$, $f(x,t)$ is a given function.

It is worth emphasizing that the study of local and non-local boundary value problems for degenerate partial differential equations is one of the rapidly advancing areas in the theory of differential equations. This growing interest is due, in part, to the fact that such degenerate equations naturally arise in the mathematical modeling of various real-world processes. At the same time, their study represents an inherent challenge within the theory of differential equations itself. For these reasons, researchers have devoted considerable attention to investigating problems involving degenerate differential equations. For further information on degenerate differential equations and related boundary value problems, we refer the reader to [19], [20], [21], [22], and the references therein.

The structure of this paper is organized as follows. Section 2 presents the formulation of the main problem. In Section 3, we establish the existence of eigenvalues and eigenfunctions for a spectral problem derived from the main problem using the method of separation of variables. Section 4 provide convergence of basic bilinear series. Section 5 examines the convergence of certain series involving eigenfunctions and Fourier coefficients. In Section 6, justification of the method of Fourier. In Section 7, we demonstrate the existence of a solution to the main problem for various values of the degeneration order. Finally, Section 8 provides our concluding remarks on the proposed approach.

II. FORMULATION OF THE PROBLEM

In the domain $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$, we consider the following equation

(5) ${}_C D_{0t}^{\alpha} u(x,t) + \left[x^{\beta} u_{xx}(x,t)\right]_{xx} = f(x,t),$

where ${}_C D_{0t}^{\alpha}$ is Caputo fractional differential operator α order [28]

$${}_C D_{0t}^{\gamma} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(z)\,dz}{(t-z)^{\gamma-n+1}}, \quad (n = [\mathrm{Re}(\gamma)] + 1,\ t > 0),$$

α, β, T are given real numbers, such that $1 < \alpha < 2$, $0 < \beta < 1$, $T > 0$; $f(x,t)$ is a given function on Ω.

First, we introduce the definition of the regular solution of the equation (5).

Definition 1. A function $u(x,t)$ satisfying in the domain Ω equation (5) and the following conditions $u(x,t)$, $u_x(x,t)$, $x^{\beta} u_{xx}(x,t)$, $\frac{\partial}{\partial x}\left(x^{\beta} u_{xx}(x,t)\right) \in C(\overline{\Omega})$, ${}_C D_{0t}^{\alpha} u(x,t)$,

$\left(x^{\beta} u_{xx}\right)_{xx} \in C(\Omega)$ is called regular in the domain Ω solution of the equation (5).

In the domain Ω, we study the following mixed problem for the equation (5):

Problem 1. *Find a regular in the domain Ω solution of the equation (5) satisfying the following initial*

(6) $u(x,0) = \varphi(x), \quad 0 \le x \le 1, \qquad u_t(x,0) = \psi(x), \qquad 0 \le x \le 1,$

and the following boundary conditions:

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(7) $u(1,t)=0,\ \dfrac{\partial u}{\partial x}(0,t)=0,\ \dfrac{\partial u}{\partial x}(1,t)=0,\ \lim\limits_{x\to 0}\dfrac{\partial}{\partial x}\left(x^{\beta}\dfrac{\partial^2 u}{\partial x^2}\right)=0,\ 0<t\le T$,

where $\varphi(x)$, $\psi(x)$ *are a given functions continuous on* $[0,1]$, *such that* $\varphi(1)=0$, $\varphi'(0)=0$, $\varphi'(1)=0$, $\lim\limits_{x\to 0}\left[x^{\beta}\varphi''(x)\right]'=0$.

Let us consider homogeneous equation of the equation (5):

${}_C D_{0t}^{\alpha}u(x,t)=\left[x^{\beta}u_{xx}(x,t)\right]_{xx},\ (x,t)\in\Omega$.

We will seek a solution of the homogeneous equation satisfying the conditions (7) in the form $u(x,t)=v(x)T(t)$. Then, with respect to the function $T(t)$, we get the following equation

${}_C D_{0t}^{\alpha}T(t)-\lambda T(t)=0$,

and with respect to $v(x)$, we get the following spectral problem:

(8) $Lv=\left[x^{\beta}v''(x)\right]''=\lambda v(x),\ x\in(0,1)$;

(9) $v(1)=0,\ v'(0)=0,\ v'(1)=0,\ \lim\limits_{x\to 0}\dfrac{\partial}{\partial x}\left(x^{\beta}\dfrac{\partial^2 v}{\partial x^2}\right)=0$,

i.e. a problem finding those values of the parameter λ for which there exist nontrivial solution of equation (8) satisfying conditions (9).

III. STUDY OF THE SPECTRAL POROBLEM

Now, we study spectral problem {(8),(9)}. Assume that there exists eigenvalues of the spectral problem {(8),(9)}. Under this assumption, first, we will define the sign of the eigenvalue λ. To this end, by multiplying both sides of (8) to the function $v(x)$ and integrating on $[0,1]$, we obtain

$$\lambda\int_0^1 v^2(x)dx=\int_0^1\left[x^{\beta}v''(x)\right]''v(x)dx.$$

Using the rule of integration by parts two times on the right-hand side of the last equality, we have

$$\lambda\int_0^1 v^2(x)dx=\left[\left[x^{\beta}v''(x)\right]'v(x)-x^{\beta}v''(x)v'(x)\right]\Big|_0^1+\int_0^1 x^{\beta}\left[v''(x)\right]^2dx.$$

Hence, considering (9), we get the following equality

$$\lambda\int_0^1 v^2(x)dx=\int_0^1 x^{\beta}\left[v''(x)\right]^2dx,$$

which implies that $\lambda\ge 0$.

Let $\lambda=0$, i.e. consider the equation $\left[x^{\beta}v''(x)\right]''=0$. It is easy to see that the general solution of this equation has the form

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$$v(x)=C_1\frac{x^{3-\beta}}{(2-\beta)(3-\beta)}+C_2\frac{x^{2-\beta}}{(1-\beta)(2-\beta)}+C_3x+C_4,$$

where C_j, $j=\overline{1,4}$ are arbitrary constants.

By obeying this function to the condition (9), we get

$$C_1=0,\ C_3=0,\ \frac{C_2}{1-\beta}=0,\ \frac{C_2}{(1-\beta)(2-\beta)}+C_4=0,$$

from which it follows that $C_1=C_2=C_3=C_4=0$. Hence, $v(x)\equiv 0$.

Thus, the problem {(8), (9)} can have nontrivial solution only for positive values of λ.

Now, we will prove the existence of eigenvalues and eigenfunctions of the spectral problem {(8), (9)}. Assuming the right-hand side of (8) is temporarily known function, we construct the Green's function of the problem {(8), (9)}. It is unique and represented as follows

$$G(x,s)=\begin{cases}\dfrac{s^{3-\beta}}{(2-\beta)(3-\beta)}-\dfrac{s^{2-\beta}x^{2-\beta}}{(1-\beta)(2-\beta)^2}+\dfrac{x^{2-\beta}s}{(1-\beta)(2-\beta)}- & \\ -\dfrac{s^{2-\beta}}{(2-\beta)^2}-\dfrac{x^{2-\beta}}{(2-\beta)^2}+\dfrac{1}{(2-\beta)^2(3-\beta)}, & x<s,\\ \dfrac{x^{3-\beta}}{(2-\beta)(3-\beta)}-\dfrac{x^{2-\beta}s^{2-\beta}}{(1-\beta)(2-\beta)^2}+\dfrac{s^{2-\beta}x}{(1-\beta)(2-\beta)}- & \\ -\dfrac{x^{2-\beta}}{(2-\beta)^2}-\dfrac{s^{2-\beta}}{(2-\beta)^2}+\dfrac{1}{(2-\beta)^2(3-\beta)}, & x>s.\end{cases}$$

Then, using Hilbert's theorem, the problem {(8), (9)} is equivalently reduced to the following integral equation [27]

(10) $v(x)=\lambda\int_0^1 G(x,s)v(s)ds$.

Since $G(x,s)$ is symmetric and continuous kernel, then from the theory of integral equations with symmetric kernels [27] it follows that equation (10) [hence problem {(8),(9)} has a countable set of eigenvalues

$$0<\lambda_1<\lambda_2<\lambda_3<...$$

and corresponding orthonormal eigenfunctions

$$v_1(x),v_2(x),v_3(x),...,$$

moreover, any arbitrary function $g(x)\in L_2[0,1]$ expands into a series in these eigenfunctions, which converges on average.

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IV. CONVERGENCE OF BASIC BILLINEAR SERIES

Let $\{v_k(x)\}_{k=1}^{\infty}$ and $\{v_k(x)\}_{k=1}^{\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(8),(9)}, respectively.

In this section, we will prove the convergence of some functional series involving these eigenvalues and eigenfunctions.

Lemma 1. *On the interval* $[0,1]$ *the following series converge uniformly:*

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2},\ \sum_{k=1}^{+\infty}\frac{\left\{[x^{\beta}v_k''(x)]'\right\}^2}{\lambda_k^2}.$$

Proof. Since, for $0<\beta<1$ the kernel $G(x,s)$ of the integral equation (10) is symmetric, continuous and positive-definite, then based on Mercer's theorem [27], we have

$$G(x,s)=\sum_{k=1}^{+\infty}\frac{v_k(x)v_k(s)}{\lambda_k},$$

particularly

$$(11)\ \sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}=G(x,x)\le const.$$

Hence, it follows that the series $\sum_{k=1}^{+\infty}\left[v_k^2(x)/\lambda_k\right]$ converges uniformly.

Now, from the integral equation (10), considering continuity of the function $G(x,s)$, we obtain

$$v_k'(x)=\lambda_k\int_0^1\frac{\partial G(x,s)}{\partial x}v_k(s)ds$$

or

$$v_k'(x)=\lambda_k\int_0^x\frac{\partial G(x,s)}{\partial x}v_k(s)ds+\lambda_k\int_x^1\frac{\partial G(x,s)}{\partial x}v_k(s)ds.$$

Using the equation (8), from the last, we have

$$v_k'(x)=\int_0^x\frac{\partial G(x,s)}{\partial x}\left[s^{\beta}v_k''(s)\right]''ds+\int_x^1\frac{\partial G(x,s)}{\partial x}\left[s^{\beta}v_k''(s)\right]''ds.$$

Hence, applying the rule of integrating by parts two times, we get

$$v_k'(x)=\left[\frac{\partial G(x,s)}{\partial x}\left[s^{\beta}v_k''(s)\right]'-\frac{\partial^2 G(x,s)}{\partial x\partial s}s^{\beta}v_k''(s)\right]_{s=0}^{s=x}+$$

$$+\left[\frac{\partial G(x,s)}{\partial x}\left[s^{\beta}v_k''(s)\right]'-\frac{\partial^2 G(x,s)}{\partial x\partial s}s^{\beta}v_k''(s)\right]_{s=x}^{s=1}+\int_0^1\frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^{\beta}v_k''(s)ds.$$

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Taking $\lim\limits_{s\to 0}\left[s^{\beta}v_k''(s)\right]'=0$, $\dfrac{\partial^2 G(x,0)}{\partial x\partial s}=0$, $\dfrac{\partial^2 G(x,1)}{\partial x\partial s}=0$, $\dfrac{\partial G(x,1)}{\partial x}=0$ and the continuity of the functions $\dfrac{\partial G(x,s)}{\partial x}$, $\dfrac{\partial^2 G(x,s)}{\partial x\partial s}$ for $x=s$, $v_k''(s)$, $\left[s^{\beta}v_k''(s)\right]'$ into account, we rewrite the last equality in the following form

$$v_k'(x)=\int_0^1 \frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^{\beta}v_k''(s)ds=\sqrt{\lambda_k}\int_0^1\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds$$

or

$$\frac{v_k'(x)}{\sqrt{\lambda_k}}=\int_0^1\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds. \tag{12}$$

It is easy to show that $\left\{s^{\beta/2}v_k''(s)/\sqrt{\lambda_k}\right\}\Big|_{k=1}^{+\infty}$ forms an orthonormal system in $L_2(0,1)$. Indeed,

$$\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^1 s^{\beta/2}v_k''(s)\cdot s^{\beta/2}v_m''(s)ds=\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^1 s^{\beta}v_k''(s)v_m''(s)ds=$$

$$=\left(\lambda_k\lambda_m\right)^{-1/2}\left\{s^{\beta}v_k''(s)v_m''(s)-\left[s^{\beta}v_k''(s)\right]'v_m(s)\right\}\Big|_0^1+\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^1\left[s^{\beta}v_k''(s)\right]''v_m(s)ds=$$

$$=\sqrt{\lambda_k/\lambda_m}\int_0^1 v_k(s)v_m(s)ds=\begin{cases}0, & for\ k\neq m,\\ 1, & for\ \ k=m.\end{cases}.$$

Then, from (12), it follows that the function $v_k'(x)/\sqrt{\lambda_k}$ can be considered as the Fourier coefficient of the function $s^{\beta/2}\dfrac{\partial^3 G(x,s)}{\partial x\partial s^2}$ with respect to variable s.

If $s^{\beta/2}\dfrac{\partial^3 G(x,s)}{\partial x\partial s^2}\in L_2[0,1]$ for all x, then based on Bessel's inequality, we obtain

$$\sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k}\le\int_0^1 s^{\beta}\left[\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]^2 ds. \tag{13}$$

Using the representation of the function $G(x,s)$, we will show that the right-hand part of the (13) is bounded. Indeed, we have

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$$\int_0^1 s^\beta \left[\frac{\partial^3 G(x,s)}{\partial x \partial s^2}\right]^2 ds = \int_0^x s^\beta \left\{\frac{\partial^3}{\partial x \partial s^2}\left(\frac{x^{3-\beta}}{(2-\beta)(3-\beta)} + \frac{x s^{2-\beta}}{(1-\beta)(2-\beta)} - \frac{s^{2-\beta}x^{2-\beta}}{(2-\beta)^2(1-\beta)} - \right.\right.$$

$$\left.\left. - \frac{s^{2-\beta}}{(2-\beta)^2} - \frac{x^{2-\beta}}{(2-\beta)^2} + \frac{1}{(3-\beta)(2-\beta)^2}\right)\right\}^2 ds + \int_x^1 s^\beta \left\{\frac{\partial^3}{\partial x \partial s^2}\left(\frac{s^{3-\beta}}{(2-\beta)(3-\beta)} - \frac{x^{2-\beta}}{(2-\beta)^2} - \right.\right.$$

$$\left.\left. - \frac{s^{2-\beta}x^{2-\beta}}{(2-\beta)^2(1-\beta)} + \frac{x^{2-\beta}s}{(2-\beta)(1-\beta)}\frac{s^{2-\beta}}{(2-\beta)^2} + \frac{1}{(3-\beta)(2-\beta)^2}\right)\right\}^2 ds =$$

$$= \frac{x^{1-\beta}}{1-\beta} - \frac{x^{2-2\beta}}{1-\beta} < \frac{1}{1-\beta}.$$

From the last, we can conclude that series $\sum_{k=1}^{+\infty}\left\{\left[v'(x)\right]^2 / \lambda_k\right\}$ converges uniformly.

Since, the function $\frac{\partial G(x,s)}{\partial x}$ is continuous, from the equation (10), we get

(14) $x^\beta v_k''(x) = \lambda_k \int_0^1 x^\beta \frac{\partial^2 G(x,s)}{\partial x^2} v_k(s)\,ds$

or

$$\frac{x^\beta v_k''(x)}{\lambda_k} = \int_0^1 x^\beta \frac{\partial^2 G(x,s)}{\partial s^2} v_k(s)\,ds\,.$$

Hence, considering $\left[x^\beta v_k''(x) / \lambda_k\right]$ as the Fourier coefficient of the function $x^\beta \frac{\partial^2 G(x,s)}{\partial x^2}$, based on Bessel's inequality, we obtain

(15) $\sum_{k=1}^{+\infty} \frac{\left[x^\beta v_k''(x)\right]^2}{\lambda_k^2} \le \int_0^1 \left[x^\beta \frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds\,.$

We show that the right-hand part of the last equality is limited:

$$\int_0^1 \left[x^\beta \frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds = \int_0^x \left[x^\beta \frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds + \int_x^1 \left[x^\beta \frac{\partial^2 G(x,s)}{\partial x^2}\right]^2 ds =$$

$$= \int_0^x x^2 - \frac{2x(1-\beta)}{(2-\beta)} + \frac{(1-\beta)^2}{(2-\beta)^2} - \frac{2xs^{2-\beta}}{2-\beta} + \frac{2s^{2-\beta}(1-\beta)}{(2-\beta)^2} + \frac{s^{4-2\beta}}{(2-\beta)^2} ds +$$

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$$+\int_x^1\left(s^2-\frac{2s(1-\beta)}{2-\beta}+\frac{(1-\beta)^2}{(2-\beta)^2}-\frac{2s^{3-\beta}}{2-\beta}+\frac{2s^{2-\beta}(1-\beta)}{(2-\beta)^2}+\frac{s^{4-2\beta}}{(2-\beta)^2}\right)ds=$$

$$=\frac{2x^3}{3}-\frac{x^2(1-\beta)}{2-\beta}-\frac{2x^{4-\beta}}{(2-\beta)(3-\beta)(4-\beta)}+\frac{1}{3}-\frac{1-\beta}{2-\beta}+\left(\frac{1-\beta}{2-\beta}\right)^2-$$

$$-\frac{2}{(4-\beta)(2-\beta)}+\frac{2(1-\beta)}{(2-\beta)^2(3-\beta)}+\frac{1}{(2-\beta)^2(5-2\beta)}=$$

$$=\frac{2x^3}{3}+\frac{1}{3}+\left(\frac{1-\beta}{2-\beta}\right)^2+\frac{2(1-\beta)}{(2-\beta)^2(3-\beta)}+\frac{1}{(2-\beta)^2(5-2\beta)}<$$

$$<1+\frac{(2\beta^4-15\beta^3+43\beta^2-56\beta+28)}{(2-\beta)^2(3-\beta)(5-2\beta)}<1+\frac{2\beta^4+43\beta^3+28}{(2-\beta)^2(3-\beta)(5-2\beta)}.$$

Hence, series $\sum_{k=1}^{+\infty}\left\{\left[x^\beta v_k''(x)\right]^2/\lambda_k^2\right\}$ converges uniformly.

Similarly, from equation (9), taking into account the function $x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}$ is continuous, we get

$$\frac{\left[x^\beta v_k''(x)\right]'}{\lambda_k}=\int_0^1\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]v_k(s)ds.$$

Hence, on the basis of Bessel's equality, we have

$$(16)\ \sum_{k=1}^{+\infty}\frac{\left[\left(x^\beta v_k''(x)\right)'\right]^2}{\lambda_k^2}\le\int_0^1\left\{\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]\right\}^2 ds.$$

We show that the right hand part of (16) is limited:

$$\int_0^1\left\{\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]\right\}^2 ds=\int_0^x\left\{\frac{\partial}{\partial x}\left[\left(x-\frac{s^{2-\beta}}{2-\beta}-\frac{(1-\beta)}{2-\beta}\right)\right]\right\}^2 ds=\int_0^x 1^2 ds=x<1.$$

Hence it follows that the series $\sum_{k=1}^{+\infty}\left\{\left[x^\beta v_k''(x)\right]^2/\lambda_k^2\right\}$ converges uniformly.

V. THE ORDER OF THE FOURIER COEFFICIENTS

Lemma 1. *Suppose the following conditions hold:*

$$f(x),\ f'(x)\in C[0,1],\ x^{\beta/2}f''(x)\in L_2[0,1],\ f(1)=0,\ f'(1)=0,\ f'(0)=0.$$

Then the following Bessel type equality is valid:

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$$(17)\ \sum_{k=1}^{\infty}\lambda_k f_k^2 \le \int_0^1 x^{\beta}\left[f''(x)\right]^2 dx.$$

Particularly, it is possible to assert convergence of the series in (17).

Here in after f_k denotes Fourier coefficient of the function $f(x)$ by the system of eigenfunctions $v_k(x)$.

Proof. Let's consider the functional

$$J=\int_0^1 x^{\beta}\left\{\left[f(x)-\sum_{k=1}^{n-1} f_k v_k(x)\right]''\right\}^2 dx=\int_0^1 x^{\beta}\left[f''(x)\right]^2 dx+\int_0^1 x^{\beta}\left[\sum_{k=1}^{n-1} f_k v_k''(x)\right]^2 dx-$$

$$-2\int_0^1 x^{\beta} f''(x)\left[\sum_{k=1}^{n-1} f_k v_k''(x)\right]dx=\int_0^1 x^{\beta}\left[f''(x)\right]^2 dx+\sum_{k=1}^{n-1} f_k^2\int_0^1 x^{\beta}\left[v_k''(x)\right]^2 dx+$$

$$(18)\ +2\sum_{\substack{k,l=1\\k\neq l}}^{n-1} f_k f_l\int_0^1 x^{\beta}v_k''(x)v_l''(x)dx-2\sum_{k=1}^{n-1} f_k\int_0^1 x^{\beta}f''(x)v_k''(x)dx.$$

Integrating by parts twice, we get

$$\int_0^1 x^{\beta}v_k''(x)v_l''(x)dx=\left[x^{\beta}v_k''(x)v_l'(x)-\left[x^{\beta}v_k''(x)\right]'v_l(x)\right]_{x=0}^{x=1}+$$

$$(19)\ +\int_0^1\left[x^{\beta}v_k''(x)\right]''v_l(x)dx=\lambda_k\int_0^1 v_k(x)v_l(x)dx=\begin{cases}0 & for\ \ k\neq l,\\ \lambda_k & for\ \ k=l.\end{cases}$$

Similarly integrating by parts, we obtain

$$\int_0^1 x^{\beta}f''(x)v_k''(x)dx=\left[x^{\beta}v_k''(x)f'(x)-\left[x^{\beta}v_k''(x)\right]'f(x)\right]_{x=0}^{x=1}+$$

$$(20)\ +\int_0^1\left[x^{\beta}v_k''(x)\right]''f(x)dx=\lambda_k\int_0^1 f(x)v_k(x)dx=\lambda_k f_k.$$

By virtue of (19) and (20), from (18) it follows that

$$J=\int_0^1 x^{\beta}\left[f''(x)\right]^2 dx-\sum_{k=1}^{n-1}\lambda_k f_k^2\ge 0\ \text{ for }\ \forall n\in N.$$

From the last inequality, it follows that the fairness of the inequality (18).

Lemma 2. Suppose that the following conditions hold:

$$f(x),\ f'(x),\ x^{\beta}f''(x),\left[x^{\beta}f''(x)\right]'\in C[0,1],\left[x^{\beta}f''(x)\right]''\in L_2[0,1],$$

Then the following Bessel type equality is valid:

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(21) $\sum_{k=1}^{\infty}\lambda_k^2 f_k^2 \le \int_0^1\left[\left(x^\beta f''(x)\right)''\right]^2 dx.$

Particularly, it is possible to assert convergence of the series in (21).

Proof. By virtue of the conditions of lemma 2 equality (20) is fulfilled. In addition, the following equality is valid as

(22) $\int_0^1 x^\beta f''(x)v_k''(x)dx = \int_0^1\left[x^\beta f''(x)\right]'' v_k(x)dx.$

Indeed, integrating by parts the left hand part of the last equality gives

$$\int_0^1 x^\beta v_k''(x)\, f''(x)dx = \left\{x^\beta f''(x)v_k'(x) - \left[x^\beta f''(x)\right]' v_k(x)\right\}\Big|_{x=0}^{x=1} + +\int_0^1\left[x^\beta f''(x)\right]'' v_k(x)dx.$$

Taking into account $x^\beta f''(x)$, $\left[x^\beta f''(x)\right]'$ are continuous on $[0,1]$, $\left[x^\beta f''(x)\right]'\Big|_{x=0} = 0$, $v_k'(0)=0$, and also conditions $v_k'(1)=0$, $v_k(1)=0$, we get

$$\left[x^\beta f''(x)v_k''(x) - \left[x^\beta f_k''(x)\right]' v_k(x)\right]_{x=0}^{x=1} = 0.$$

From (16) and (18) it follows that

$$\int_0^1\left[x^\beta f''(x)\right]'' v_k(x)dx = \lambda_k f_k.$$

Hence, $\lambda_k f_k$ is the Fourier coefficient of the function $Lf \equiv \left[x^\beta f''(x)\right]''$, or

(23) $\left(Lf\right)_k = \left[\left(x^\beta f''(x)\right)''\right]_k = \lambda_k f_k.$

Writing Bessel's inequality for function $\left[x^\beta f''(x)\right]''$ and taking equality (23) into account, we get sought inequality.

VI. JUSTIFICATION OF THE METHOD OF FOURIER

Theorem 1. Suppose that the following conditions hold:

1. $\varphi(x)$ *satisfies conditions of the lemma 1.*
2. $f(x,t)$ *satisfies conditions of the lemma 1 uniformly with respect to* t
3. $f_t'(x,t)$ *satisfies conditions of the lemmas 1 and 2 uniformly with respect to variable* t.

Then the following

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$$u(x,t)=\sum_{k=1}^{+\infty}\Big[\varphi_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)+\psi_k t E_{\alpha,2}\left(\lambda_k t^{\alpha}\right)+ \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda_k (t-s)^{\alpha}\right]f_k(s)ds\Big]v_k(x) \tag{24}$$

series will be regular solution of the problem $\{(5),(6),(7)\}$ *,where* $E_{\alpha,\beta}(z)$ *is two-parameter Mittage-Leffler's function:*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \qquad \alpha,\beta>0.$$

Proof. We shall seek solution of the considered in the form

$$u(x,t)=\sum_{k=1}^{\infty}T_k(t)v_k(x) \tag{25}$$

where $v_k(x),\ k\in N$ are functions which are defined by (11), and $T_k(t),\ k\in N$ are unknown functions.

Expanding functions $f(x,t)$ and $\varphi(x),\ \psi(x)$ to series by functions $v_k(x)$, we get

$$f(x,t)=\sum_{k=1}^{\infty}f_k(t)v_k(x),\ \varphi(x)=\sum_{k=1}^{\infty}\varphi_k v_k(x),\ \psi(x)=\sum_{k=1}^{\infty}\psi_k v_k(x), \tag{26}$$

where

$$f_k(t)=\int_0^1 f(x,t)v_k(x)dx,\ \varphi_k=\int_0^1\varphi(x)v_k(x)dx,\ \psi_k=\int_0^1\psi(x)v_k(x)dx. \tag{27}$$

Substituting (25) and (26) into equation (5) and the condition (6), we obtain

$$\sum_{k=1}^{+\infty}{}_C D_{0t}^{\alpha}T_k(t)\cdot v_k(x)+\sum_{k=1}^{+\infty}T_k(t)\left[x^{\beta}v_k''(x)\right]''=\sum_{n=1}^{\infty}f_k(t)v_k(x),$$

$$\sum_{k=1}^{+\infty}T_k(0)\cdot v_k(x)=\sum_{k=1}^{\infty}\varphi_k\cdot v_k(x),\quad \sum_{k=1}^{+\infty}T_k'(0)\cdot v_k(x)=\sum_{k=1}^{\infty}\varphi_k\cdot v_k(x).$$

From the last equalities, by virtue of (8) and completeness of the system of functions $v_k(x),\ k\in N$ in the space $L_2(0,1)$, it follows the following problem with respect to unknown function $T_k(t)$:

$${}_C D_{0t}^{\alpha}T_k(t)+\lambda_k T_k(t)=f_k(t),\ T_k(0)=\varphi_k,\quad T_k'(0)=\psi_k\ .$$

It is known the solution of this problem represented as follows [28]

$$T_k(t)=\varphi_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)+\psi_k t E_{\alpha,2}\left(\lambda_k t^{\alpha}\right)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda_k (t-s)^{\alpha}\right]f_k(s)ds$$

Substituting this expression of $T_k(t)$ into (25), we obtain formula (24).

Now, we show uniformly convergence of the series

$$u(x,t),\ u_x(x,t),\ x^{\beta}u_{xx}(x,t),\ \left[x^{\beta}u_{xx}(x,t)\right]_x,\ {}_C D_{0t}^{\alpha}u(x,t).$$

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Using the rule of integrating by parts we rewrite (24) as follows

$$u(x,t)=\sum_{k=1}^{+\infty}E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k v_k(x)+\sum_{k=1}^{+\infty}E_{\alpha,2}\left(\lambda_k t^{\alpha}\right)t\psi_k v_k(x)-$$

(28)

$$-\sum_{k=1}^{+\infty}t^{\alpha}E_{\alpha,\alpha+1}\left(\lambda_k t^{\alpha}\right)f_k(0)v_k(x)-\sum_{k=1}^{+\infty}\int_0^t(t-s)^{\alpha}E_{\alpha,\alpha+1}\left[\lambda_k(t-s)^{\alpha}\right]f_k'(s)ds\cdot v_k(x)$$

We show absolute and uniformly convergence of the series in the right hand side of (28).

Since the Mittag-Leffler function is bounded [28], i.e.

(29) $\left|E_{\alpha,\beta}(z)\right|\le M$, $0<M<+\infty$, $z>0$,

one can easily verify that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k\cdot v_k(x)\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,2}\left(\lambda_k t^{\alpha}\right)t\psi_k\cdot v_k(x)\right|\le M\cdot T\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}f_k(0)t^{\alpha}E_{\alpha,\alpha+1}\left(\lambda_k t^{\alpha}\right)v_k(x)\le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t(t-s)^{\alpha}E_{\alpha,\alpha+1}\left[\lambda_k(t-s)^{\alpha}\right]f_k'(s)ds\cdot v_k(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(s)ds\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

We show the validity of the first inequality. Using (29) and Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k\cdot v_k(x)\right|\le M\sum_{k=1}^{+\infty}\left|\varphi_k\cdot v_k(x)\right|=$$

$$=M\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k}\varphi_k\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\psi_k\cdot v_k(x)\right|\le M\cdot T\sum_{k=1}^{+\infty}\left|\psi_k\cdot v_k(x)\right|=$$

$$=M\cdot T\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k}\psi_k\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le M\cdot T\left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}$$

Similarly, one can show validity of the third and fourth inequalities.

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From these inequalities, taking into account lemma 1 and lemma 2, it follows that absolutely and uniformly convergence of the series of (28) and $u(x,t) \in C(\bar{\Omega})$.

Differentiating (28) with respect to x, we obtain

$$u_x(x,t) = \sum_{k=1}^{+\infty} E_{\alpha,1}(\lambda_k t^\alpha)\varphi_k v_k'(x) + \sum_{k=1}^{+\infty} E_{\alpha,1}(\lambda_k t^\alpha) t\psi_k v_k'(x) -$$

$$-\sum_{k=1}^{+\infty} E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) t^\alpha f_k(0) v_k'(x) - \sum_{k=1}^{+\infty} \int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}\left[\lambda_k (t-s)^\alpha\right] f_k'(s)ds \cdot v_k'(x) \quad (30)$$

Analogously, it can be shown that the following inequalities hold:

$$\sum_{k=1}^{+\infty} \left|E_{\alpha,1}(\lambda_k t^\alpha)\varphi_k v_k'(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2 \cdot \sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty} \left|E_{\alpha,1}(\lambda_k t^\alpha) t\psi_k v_k'(x)\right| \le M\cdot T\left[\sum_{k=1}^{+\infty}\lambda_k\psi_k^2 \cdot \sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty} \left|E_{\alpha,1}(\lambda_k t^\alpha) t^\alpha f_k(0) v_k'(x)\right| \le M\cdot T^\alpha\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0) \cdot \sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty} \int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}\left[\lambda_k (t-s)^\alpha\right] f_k'(s)ds \cdot v_k'(x) \le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T \sum_{k=1}^{+\infty}\lambda_k f_k'^2(s)ds \cdot \sum_{k=1}^{+\infty}\frac{v_k'^2(x)}{\lambda_k}\right]^{1/2}.$$

From the last inequalities by virtue of the conditions of Theorem 1, it follows that the absolute and uniformly convergence of the series in (30).

Now, differentiating (30) with respect to x and multiplying the both sides of the taken equality to x^β, we obtain

$$x^\beta u_{xx}(x,t) = \sum_{k=1}^{+\infty} E_{\alpha,1}(\lambda_k t^\alpha)\varphi_k x^\beta v_k''(x) + \sum_{k=1}^{+\infty} E_{\alpha,1}(\lambda_k t^\alpha) t\psi_k x^\beta v_k''(x) -$$

$$\sum_{k=1}^{+\infty} E_{\alpha,\alpha+1}(\lambda_k t^\alpha) t^\alpha f_k(0) x^\beta v_k''(x) - \sum_{k=1}^{+\infty} \int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}\left[\lambda_k (t-s)^\alpha\right] f_k'(s)ds \cdot x^\beta v_k''(x) \quad (31)$$

We show the convergence of the series in (31). Firstly, we show that the following inequalities are valid:

$$\sum_{k=1}^{+\infty} \left|E_{\alpha,1}(\lambda_k t^\alpha)\varphi_k x^\beta v_k''(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_k^2 \cdot \sum_{k=1}^{+\infty}\frac{\left[x^\beta v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

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$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)t\psi_k x^{\beta}v_k''(x)\right| \le M\cdot T\left[\sum_{k=1}^{+\infty}\lambda_k^2\psi_k^2\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)t^{\alpha}f_k(0)x^{\beta}v_k''(x)\right| \le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k^2 f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t(t-s)^{\alpha}E_{\alpha,\alpha+1}\left[\lambda_k(t-s)^{\alpha}\right]f_k'(s)ds\cdot x^{\beta}v_k''(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2 f_k'^2(s)ds\cdot\sum_{k=1}^{+\infty}\frac{x^{2\beta}v_k''^2(x)}{\lambda_k^2}\right]^{1/2}.$$

Let's prove the first inequality. Taking into account (29), we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)\right| \le M\sum_{k=1}^{+\infty}\left|\varphi_k x^{\beta}v_k''(x)\right| = M\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_k\cdot\frac{x^{\beta}v_k''(x)}{\lambda_k}\right|.$$

Applying Cauchy-Schwarz inequality from the last, we obtain proof of the first inequality. The third and fourth inequalities are proved similarly.

Using the same method one can prove absolute and uniformly convergence of the series in $\left[x^{\beta}u_{xx}(x,t)\right]_x$.

Now, we investigate ${}_C D_{0t}^{\alpha}u(x,t)$. Applying differential operator ${}_C D_{0t}^{\alpha}$ to the both sides of (28) and taking into account ${}_C D_{0t}^{\alpha}T_k(t) = -\lambda_k T_k(t)$ and also the view of the function $T_k(t)$, we find

$${}_C D_{0t}^{\alpha}u(x,t) = -\sum_{k=1}^{+\infty}\lambda_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k v_k(x) - \sum_{k=1}^{+\infty}\lambda_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)t\psi_k v_k(x) +$$

$$+\sum_{k=1}^{+\infty}\lambda_k E_{\alpha,\alpha+1}\left(\lambda_k t^{\alpha}\right)t^{\alpha}f_k(0)v_k(x) + \sum_{k=1}^{+\infty}\lambda_k\int_0^t(t-s)^{\alpha}E_{\alpha,\alpha+1}\left[\lambda_k(t-s)^{\alpha}\right]f_k'(s)ds\cdot v_k(x)$$

And convergence of the series in ${}_C D_{0t}^{\alpha}u(x,t)$ follows by the following inequalities

$$\sum_{k=1}^{+\infty}\left|\lambda_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)\varphi_k v_k(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)t\psi_k v_k(x)\right| \le M\cdot T\left[\sum_{k=1}^{+\infty}\lambda_k^3\psi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|\lambda_k E_{\alpha,1}\left(\lambda_k t^{\alpha}\right)t^{\alpha}f_k(0)v_k(x)\right| \le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

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$$\sum_{k=1}^{+\infty}\left|\lambda_k\int_0^t(t-s)^{\alpha}E_{\alpha,\alpha+1}\left[\lambda_k(t-s)^{\alpha}\right]f_k'(s)ds\cdot v_k(x)\right|\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^3 f_k'^2(s)ds\cdot\sum_{k=1}^{+\infty}\frac{v^2(x)}{\lambda_k}\right]^{1/2}.$$

These inequalities are proved the same method as above proved inequalities. From these inequalities, it follows the absolute and uniform convergence of the series in ${}_C D_{0t}^{\alpha}u(x,t)$.

Theorem 1 has been proved.

VII. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

Now, we prove the uniqueness of the solution of the problem. For this aim, we introduce the following function

(32) $T_k(t)=\int_0^1 u(x,t)v_k(x)dx.$

Based on (31), we consider the following auxiliary function

(33) $T_{k,\varepsilon}(t)=\int_{\varepsilon}^{1-\varepsilon}u(x,t)v_k(x)dx,$

where ε is sufficiently small positive number.

Applying differential operator ${}_C D_{0t}^{\alpha}$ to (33) and using homogeneous equation corresponding equation (5), we get

$${}_C D_{0t}^{\alpha}T_{k,\varepsilon}(t)=-\int_{\varepsilon}^{1-\varepsilon}\left[x^{\beta}u_{xx}(x,t)\right]_{xx}\cdot v_k(x)dx.$$

Using the rule of integration by parts four times from the last equality, we get

$${}_C D_{0t}^{\alpha}T_{k,\varepsilon}(t)=-\Big\{v_k(x)\left[x^{\beta}u_{xx}(x,t)\right]_x-v_k'(x)x^{\beta}u_{xx}(x,t)+x^{\beta}v_k''(x)u_x(x,t)-$$

(34) $$-\left[x^{\beta}v_k''(x)\right]'u(x,t)\Big\}\Big|_{x=\varepsilon}^{x=1-\varepsilon}-\int_{\varepsilon}^{1-\varepsilon}\left[x^{\beta}v_k''(x)\right]''u(x,t)dx.$$

Passing to the limit as $\varepsilon\to 0$ and taking (9) and (32) into account, from (34), we derive

(35) ${}_C D_{0t}^{\alpha}T_k(t)+\lambda_k T_k(t)=0.$

From (32), we have

(36)
$$T_k(0)=\int_0^1 u(x,0)v_k(x)dx=\varphi_k$$
$$T_k'(0)=\int_0^1 u'(x,0)v_k(x)dx=\psi_k$$

It is known that, the solution of the equation (35) satisfying condition (36) is represented as follows

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$T_k(t) = \varphi_k E_{\alpha,1}(\lambda_k t^\alpha) + \psi_k t E_{\alpha,2}(\lambda_k t^\alpha)$.

Let $\varphi(x) \equiv 0$, $x \in [0,l]$. Then from the last equality taking into account (32) for all $t \in [0,\mathrm{T}]$ and $k \in N$ it follows that

(37) $\int_0^1 u(x,t) v_k(x) dx = 0$.

As since problem {(8),(9)} is self-adjoint, its eigenfunctions will be complete system in $L_2[0,l]$. Taking this account from (37), we get $u(x,t) \equiv 0$ almost everywhere on $[0,l]$ for all $t \in [0,T]$. By virtue of $u(x,t) \in C(\bar{\Omega})$, we obtain $u(x,t) \equiv 0$ in $\bar{\Omega}$. Thus, homogeneous problem has only trivial solution and this gives us the uniqueness of the solution of the considered problem.

VIII. CONCLUSION

In the present paper, we consider an initial-boundary value problem for a fourth-order, time-fractional, space-degenerate partial differential equation in a rectangular domain. By applying the method of separation of variables, we obtain a spectral problem for an ordinary differential equation. Since this equation has a degenerate coefficient, we cannot find the eigenvalues and eigenfunctions of this spectral problem in explicit form. However, by applying the theory of integral equations with symmetric kernels, we establish the existence and some properties of the eigenvalues and eigenfunctions of this spectral problem. Using these properties, we prove the existence and uniqueness of the main problem.

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